

MEMORANDUM

RM-4815-NASA

DECEMBER 1965

NAVIGATION BY SATELLITE USING
TWO-WAY RANGE AND DOPPLER DATA

J. H. Hutcheson, J. D. Mallett and H. S. Schwimmer

This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

The **RAND** *Corporation*

1700 MAIN ST • SANTA MONICA • CALIFORNIA • 90406

PREFACE

At the present time there is a need for expanding traffic control and navigation facilities, particularly for ships and transoceanic aircraft. Moreover, increased growth of transoceanic travel and increased speeds will compound the problem and require greatly increased communication capability. It seems almost certain that satellites will have a definite role in future communication, traffic control and coordination. This Memorandum concerns an investigation of a method for obtaining a navigation capability as a secondary or bonus feature to a communication satellite system.

So far as the authors are aware, the use of range and range-rate data in combination represents a different approach to the communication, control and navigation problem and has certain novel and attractive features which warrant investigation. The results of this preliminary work should prove useful to others who may wish to extend the studies further, if low-altitude (< 4000 n mi) communications satellites prove desirable.

This Memorandum was prepared under the Communications Satellite Technology contract with the National Aeronautics and Space Administration. It was undertaken at the informal request of Mr. Greg Andrus, Advanced Technology Manager of the Communications and Navigation Programs Division.

SUMMARY

The primary motivation for this study was the belief that a navigation capability should be developed as a secondary or bonus feature to a communication satellite system since the primary need is for coordination, control and communication. With this objective in mind, this Memorandum is concerned with the technology involved in determining a user vehicle's position from a single satellite by obtaining multiple measurements of range and range-rate data. The vehicle can be anything from a submarine to a supersonic transport. The computational method is based on a "six-element fix" where three unknown position components and three unknown velocity components are determined from a set of measurements of range and range-rate data. In principle, it is an orbit determination process in reverse. A simplified model of the process was developed and tested on a digital computer in order to test for convergence and sensitivities to error when the time interval between measurements was decreased. Many examples were simulated with vehicle speeds ranging from 20 K for surface vessels to 2000 K for supersonic aircraft and satellite altitudes ranging from 500 n mi to synchronous, both polar and equatorial. The results, although by no means complete, are sufficient to indicate that satisfactory aircraft navigation accuracies may be achieved for orbital altitudes of less than 4000 n mi. At the higher altitudes, the navigation process is too sensitive to measurement errors in doppler shift using measurement time intervals which would be practical for high-speed aircraft. The results of this investigation indicate that further study efforts on these techniques may be warranted, if low-altitude communications satellites become available.

CONTENTS

PREFACE	iii
SUMMARY	v
LIST OF FIGURES	ix
LIST OF SYMBOLS	xi
Section	
I. INTRODUCTION	1
II. SATELLITE NAVIGATION SCHEMES	3
III. USE OF RANGE AND DOPPLER DATA	5
IV. EQUIPMENT ACCURACY	11
V. FIX DETERMINATION PROCESS	23
VI. SIMULATION RESULTS	28
VII. CONCLUSIONS	37
Appendix	
A. SIMPLIFIED THREE-DIMENSIONAL ERROR ANALYSIS	39
B. FORMULAS APPLYING TO FIX DETERMINATION PROCESS	44
C. EVALUATION OF PARTIAL DERIVATIVES	49
REFERENCES	60

LIST OF FIGURES

1. Conical and spherical surface of position	7
2. Solution by matrix inversion	30
3. Effects of time interval on convergence (error free measurements)	32
4. Navigation errors due to $\Delta \dot{r}$ measurement error, polar orbits	33
5. Effects of measurement time interval on navigation error	35
6. Navigation error due to $\Delta \dot{r}$ measurement error, equatorial and polar orbits	36

LIST OF SYMBOLS

A	course line angle measured clockwise from north; also, effective antenna aperture
a	semimajor axis of an elliptical satellite orbit (equal to the orbital radius for a circular orbit)
a_{ij}	the ij -th element of a matrix
B	signal bandwidth
c	speed of light
E	eccentric anomaly given by solution of Kepler's equation; also, transmitted energy
e	orbital eccentricity
e/n	signal energy-to-noise ratio
f	frequency
f_d	doppler shift of frequency f due to speed v
G	antenna gain
h	altitude above earth's surface
\dot{h}	time rate of change of altitude
i	inclination of the orbital plane
k	Boltzmann's constant
k_1	constant dependent on energy distribution
M	matrix of partial derivative coefficients
n	mean angular rate of the satellite
n_E	earth's rotational rate
P	magnitude of vehicle position vector
$P(r)$	component of \bar{P} in the \bar{r} direction
P_x, P_y, P_z	components of \bar{P} in the x, y, z direction re- spectively

$P(\psi)$	component of \bar{P} in the ψ direction
\bar{P}	vehicle position vector directed from the geocenter to the vehicle
$\dot{\bar{P}}$	time rate of change of vehicle position vector with respect to inertial space
\hat{P}	peak power
P_x, P_y, P_z	components of \bar{p} in the x, y, z direction respectively
\bar{p}	unit vector along line of apsides directed toward perigee
q_i	generalized parameter where $i = 1, 2, 3, 4$ and $q_1 = \text{latitude}$, $q_2 = \text{longitude}$, $q_3 = \text{heading}$ and $q_4 = \text{speed}$, all considered at time t_1
q_x, q_y, q_z	components of \bar{q} in x, y, z direction respectively
\bar{q}	unit vector normal to \bar{p} and in direction of tangential velocity
R_e	radius of the earth
r	range, magnitude of range vector
r_b	range between transmitted pulses
r_u	unambiguous range
r_x, r_y, r_z	components of range in the x, y, z direction respectively
\bar{r}	range vector measured from \bar{S} to \bar{P}
\dot{r}	range rate, magnitude of range rate
$\dot{\bar{r}}$	time rate of change of range vector \bar{r} with respect to inertial space
S	magnitude of satellite position vector
S_x, S_y, S_z	components of satellite position in x, y, z direction respectively
\bar{S}	satellite position vector directed from the geocenter to the satellite

$\dot{\bar{S}}$	time rate of change of the satellite position vector with respect to inertial space
s	signal amplitude
T	time to make a velocity measurement, i.e., doppler shift; also, time of perigee passage
T_{eff}	effective temperature
t	universal time
V	speed of vehicle
V_p	magnitude of vehicle velocity vector
V_{px}, V_{py}, V_{pz}	components of the vehicle velocity in the x, y, z direction respectively
V_r	magnitude of relative velocity vector \bar{V}_r
V_{rx}, V_{ry}, V_{rz}	components of relative velocity in the x, y, z direction respectively
\bar{V}_p	vehicle velocity vector
\bar{V}_r	relative velocity vector, satellite to vehicle (expressed in an earth-fixed rotating coordinate system)
v	relative speed between the transmitting source and the receiver
x_w	component of \bar{S} in orbital plane (x_w -axis from focus to perigee)
y_w	component of \bar{S} in orbital plane (y_w -axis along semilatus rectum)
γ	angle between the vectors \bar{P} and \bar{S}
Δa	vector of range and range-rate residuals
Δp	vector of vehicle position and velocity error
θ	angle from the vernal equinox to the object of interest
λ	electrical wave length; also, longitude
τ	total (two-way) transit time, i.e., time interval between transmission of a pulse and the reception of its echo

ψ	angle between the relative velocity \bar{V}_r and the range vector
Ω	node angle
ω	argument of perigee

I. INTRODUCTION

The RAND Corporation has been working for NASA Headquarters under Contract NASr-21(02) in connection with future requirements for non-military traffic coordination, control and navigation systems. This work has been prompted by the fact that there is an increasing need for frequent and accurate position information about ships and transoceanic aircraft for use by traffic coordination agencies and in search and rescue operations. This need is only partially satisfied by existing systems. There is little doubt that the forecasted growth of transoceanic travel and greatly increased aircraft speeds will compound the problem and impose an enormous burden on communication facilities. Moreover, there is little reason to believe that there is a one-to-one relationship between the expected transportation growth and the accompanying communication needs. It is most likely that the bandwidth requirements per vehicle will greatly increase the more affluent and technological our society becomes. Present transoceanic HF communication channels are already overloaded during peak traffic periods and will reach saturation within a few years unless effective improvements are instituted. Even with more efficient utilization and advanced technology, it is doubtful if there is sufficient growth potential in the HF, VHF, or currently used UHF portions of the spectrum to keep pace with the worldwide communication needs. Looking to the future, it seems clear that communication satellites operating at microwave frequencies will be used to alleviate the problem. As for the navigation function inherent in traffic control, the Technology Audit Corporation in a recent study⁽¹⁾ stated that there is "no urgent need for navigation systems in

addition to those presently available or projected" (i.e., Loran C, Omega, Delrac). But there is an urgent need for an advisory relationship between shore agencies and the nonmilitary navigator for collision hazard information, traffic control, search and rescue, etc. It is probable that as oceanic traffic density increases, any craft which is potentially a hazard to another will be required to come under traffic control supervision and hence carry approved equipment. Position reports must be systematically and periodically obtained and processed with high reliability (as contrasted with the present marine system which is voluntary and has about 50 percent participation). There are, therefore, certain obvious advantages from the standpoints of reliability, accuracy, and for emergencies or distress for the navigation fix to be determined and computed by a ground station rather than on board the user vehicle. This is not to rule out the possibility of making on-board computation and fix determination automatic and subject to readout upon interrogation by the control agency.

In view of the preceding discussion, it appears likely that satellites will have a definite role in future oceanic communication and traffic control and coordination. Present oceanic navigation facilities (i.e., hyperbolic systems such as Loran C) would require considerable expansion to provide global coverage with good accuracy and reliability.

It therefore seems worthwhile to consider how a navigation capability can be developed as a secondary or bonus feature to a communication satellite system. With this objective in mind, this Memorandum is concerned with a computational method for determining a user vehicle's position (navigation fix) from a single satellite by obtaining multiple measurements of range and range-rate data.

II. SATELLITE NAVIGATION SCHEMES

There are a great number of possible schemes for determining a vehicle's position from satellite measurements. It is possible to conceive of systems employing various combinations of range, range rate, angle, or angular-rate data using one satellite or several. There are also schemes (dependent or independent of the user vehicle itself) which supply certain data (e.g., altitude, speed, or course) to a precision which may vary from a crude estimate to an accurately measured quantity. These various schemes may be divided into two basic categories: one where all measurements are nearly simultaneous in time, and the other where significant time intervals occur between measurements. There is an important difference between the two since the latter involves the determination of position and velocity. For obvious reasons there are definite advantages to the simultaneous measurement method, the most important being the elimination of error due to changes in velocity during the measurement time interval. However, there is a cost in terms of complexity, reliability, number of satellites in orbits, etc., that may cause the satellite system to no longer appear to provide navigation capability merely as a secondary feature. References 2 to 5 contain detailed studies of various schemes of the simultaneous type.

The method described here is based on a "six-element fix" where a small but significant time interval exists between data measurements. In principle it is nothing more than an orbit determination process in reverse, using differential correction techniques. That is, instead of an unknown orbit with reference ground observing stations, assume

the orbit is known, and the ground station (e.g., an aircraft) is on some unknown trajectory. The problem is to find the six "elements," three for position and three for velocity. In a practical case, two of the elements are probably already known with sufficient precision, i.e., altitude and its time rate of change. This knowledge can be used to reduce the computation or perhaps as a self-checking feature. As with an orbit determination process, the computation must start with certain initial assumptions which are refined step by step as the process continues. In the reverse process of determining, say, an aircraft's flight path, it is necessary to start with a set of equations which will represent the time history of the path between observations, e.g., constant speed, constant heading (rhumb line) or great-circle course. Unfortunately, in this process there will be no prior knowledge of perturbing influences such as sudden changes in heading or speed. On the other hand, perturbing effects become small as the time interval between observations decreases; moreover, except when near terminals, most aircraft fly rather steady flight paths, particularly when navigating across oceans. Supersonic transports may travel as fast as 30 miles per minute which means that position uncertainty may accumulate quickly due to small instrumentation errors. Accordingly, it is estimated that a system should be able to provide a fix about every five minutes or less. The important question is whether or not an orbit, or, as used here, a trajectory determination process, is convergent with such short time intervals between observations. This will depend upon many factors, but most importantly upon the geometry and relative velocity between satellite and vehicle.

III. USE OF RANGE AND DOPPLER DATA

Generally speaking, an orbit, or a trajectory, may be determined by obtaining six independent observational quantities involving either range, range rate, angular data, or combinations thereof, if the equations governing the motion are known and if there are no unknown forces or accelerations. In the restricted case of a surface vessel or an aircraft at constant known altitude, only four measurements are required. Again, it is important to clarify the difference between obtaining a position at a specific time, using three simultaneous measurements, and obtaining a trajectory using nonsimultaneous measurements. In the latter case, three components of velocity must also be determined in order to obtain the desired position information because of the time interval. Furthermore, these components of velocity are treated as unknown rather than known quantities.

The use of range and/or range-rate data as observable quantities was chosen for investigation because of the precision which can be achieved without the use of overly complex hardware on either the satellite or user vehicle.

For a six-element fix, it is theoretically feasible to use either six independent measurements of range or range rate, or a mixture of the two. Results from similar past efforts (see Ref. 6) were pessimistic with regard to using a set of range-rate only measurements. Unsatisfactory results were obtained recently using a set of range-only measurements, due to the computational difficulties arising when "small" time intervals between measurements are involved. The question of "how small" is a function of vehicle and satellite speed and direction,

i.e., relative velocity. As previously discussed, intervals of less than five minutes are desirable, not only to reduce errors but for operational reasons. Note that lines of position obtained from a single satellite will intersect at very small angles for closely spaced range measurements. An attempt was made to determine how well a fix could be obtained under such conditions by using precision range measurements and modern data processing. Results using range-only measurements were rather disappointing for reasons to be discussed in Section VI which deals with the computation process. Using a mixture of range and range-rate data led to more successful results since the indeterminacy and computational difficulties were considerably lessened. This could be surmised by considering the following facts: Range rate is the component of relative velocity (vector difference between satellite and vehicle velocity) which is parallel to or along the line between satellite and vehicle. It is therefore a function (i.e., $\cos \psi$) of the angle between this relative velocity and the range vector. Since the satellite velocity is usually far greater than the vehicle velocity, the angle ψ is approximately that of the angle between the satellite velocity and the range vector. Therefore, the measurement of range rate is analogous to an angular measurement. As a simple example, consider a vehicle to be motionless on a nonrotating earth. If the satellite velocity is known, a single measurement of range rate \dot{r} will define $\cos \psi$. As illustrated in Fig. 1 below, the angle ψ defines a conical surface of position containing all points where this particular value of \dot{r} could be measured. In this simple case, the axis of the cone is along the satellite velocity vector, whereas generally the axis lies

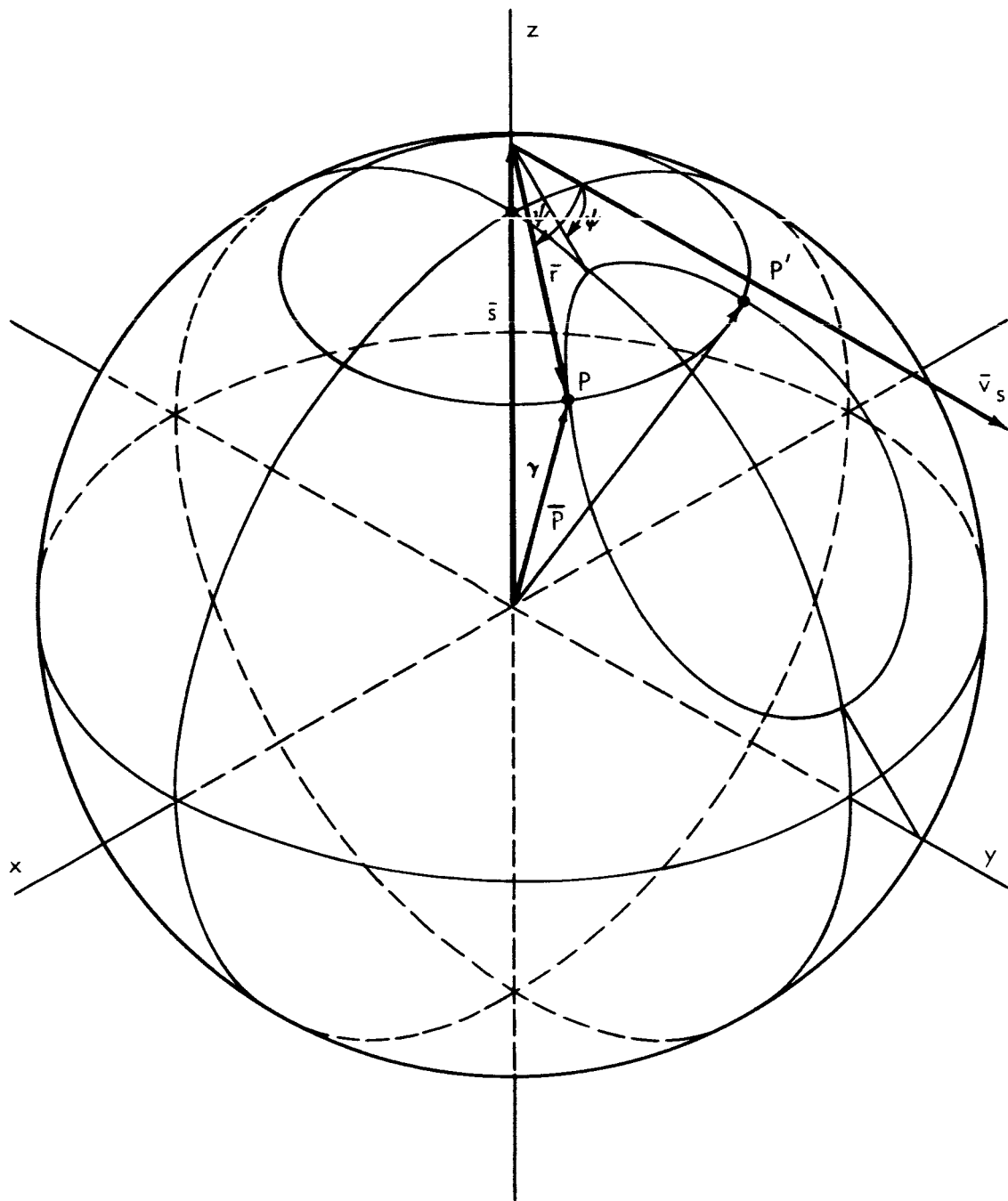
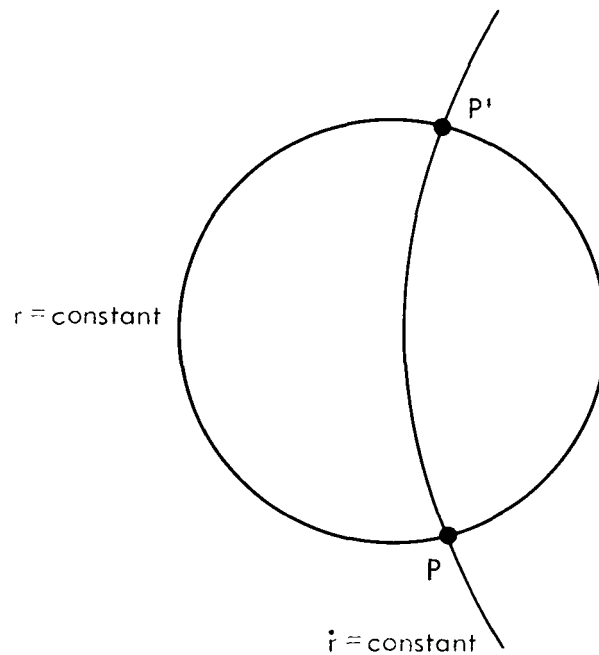


Fig.1—Conical and spherical surfaces of position

along the relative (satellite-vehicle) velocity vector. The measurement of range leads to a conical and spherical surface of position. (See Fig. 1.) Therefore, the intersection of the cone and sphere with a sphere representing the earth's surface (or a surface defined by some specified altitude) will result in two intersecting arcs of position, as shown below. The ambiguity is easily resolved with only primitive knowledge of the vehicle's position.



Intersecting arcs of position

As might be expected, the fix accuracy will be very sensitive to the determination of the angle ψ , and hence the measurement accuracy of \dot{r} . For ψ near 90° the navigation error is approximately $r\Delta\psi$, where

$$\Delta\psi \approx -\frac{\Delta\dot{r}}{V_r} \quad (1)$$

In the above, r is the range from satellite to vehicle, V_r is the magnitude of the relative velocity vector, satellite to vehicle (as seen

in an earth-fixed reference frame), and $\Delta \dot{r}$ is the measurement error in range rate.

Referring to Fig. 1, note that the range vector \bar{r} is given by

$$\bar{r} = \bar{P} - \bar{S}$$

Designating $\dot{\bar{P}}$ and $\dot{\bar{S}}$ as the velocities of P and S with respect to an inertial nonrotating coordinate system, then the relative velocity expressed in this same system is

$$\dot{\bar{r}} = \dot{\bar{P}} - \dot{\bar{S}}$$

Designating \bar{V}_r as the velocity of S relative to P apparent to an observer in a rotating earth-fixed coordinate frame, then

$$\dot{\bar{r}} = \bar{V}_r + \bar{\omega}_e \times \bar{r}$$

where $\bar{\omega}_e$ is the angular velocity vector defining the rotation of the earth-fixed axes with respect to the inertial frame. It is important to note that since the vector result of the cross product is always normal to \bar{r}

$$\dot{\bar{r}} = \frac{\bar{r} \cdot \dot{\bar{r}}}{r} = \frac{\bar{r} \cdot \bar{V}_r}{r}$$

For two fixed points on the earth's surface, $\bar{V}_r = 0$, whereas the relative velocity of the one point with respect to the other is not equal to zero when expressed as an inertial velocity (due to the earth's rotation).

In a similar fashion, the relationship between position and range measurement errors may be obtained from the triangular relationship

$$r^2 = S^2 + P^2 - 2SP \cos \gamma$$

where γ is the angle between \bar{P} and \bar{S} .

Differentiating and setting $\Delta S = \Delta P = 0$ yields

$$P\Delta\gamma = \frac{r\Delta r}{S \sin \gamma} \quad (2)$$

which shows an unfavorable condition for small values of γ , i.e., when the satellite is nearly directly over the vehicle.

The vectors \bar{P} , \bar{S} , and \bar{r} form a plane and so far only the positional geometry in this plane has been considered. The general three-dimensional case, where the relative \bar{V}_r is not necessarily coplanar, is considered in Appendix A. It is shown there that a singularity exists when the out-of-plane component of relative velocity is zero. Conversely, errors are minimized when the satellite motion relative to the vehicle is normal to the \bar{P} , \bar{S} plane.

IV. EQUIPMENT ACCURACY

The purpose of this section is to indicate the equipment accuracy as a function of required position accuracy. Since the measurements involve only round-trip time delays, the greatest burden is placed on the measurement of time, i.e., the clock used at the transmit and receive end. So long as the initial transmitter and final receiver are at the same end, it does not matter whether these are at the navigator position P, in the satellite S, or at a ground station that is tracking the satellite and communicating with it. It is assumed that the satellite position vector \bar{S} and velocity vector \bar{V}_S are precisely known and need not enter the error analysis and that delay errors in the transponders also can be made small enough to be neglected. Therefore, required clock accuracies and the time over which the measurements must be made will be estimated. It should be noted that this navigation method uses equipment and techniques similar to those used in pulse doppler radar; therefore the art is well established.

Assume for illustration that a transmitter and receiver are at point P, while the satellite has a transponder. A navigator could transmit pulses to the satellite and receive them after a time delay τ . By comparing the received pulses with those transmitted, the range and doppler shift can be determined as in radar by

$$f_d = \frac{2v}{\lambda} \left(1 + \frac{v}{c} + \left(\frac{v}{c} \right)^2 + \dots \right)$$

where

λ = transmitter wavelength

Neglecting higher order terms in $\frac{v}{c}$

$$f_d \cong \frac{2v}{\lambda} = \frac{2vf}{c} \quad (3)$$

The measured velocity $v = \dot{r}$ is a function of the angle ψ between the relative vector velocity \bar{V}_r and the line-of-sight between P and S, as discussed in Section III.

Assuming that V_r , f_d , and λ are known or can be measured, the angle ψ is given by

$$\psi = \cos^{-1} \left(\frac{\lambda f_d}{2V_r} \right) \quad (4)$$

Knowing the angle ψ places the point P on the surface of a cone that intersects the earth in a line, as discussed in Section III. The measurement of time delay τ gives range r from the relation

$$r = \frac{c\tau}{2} \quad (5)$$

which places the point P on the surface of a sphere that intersects the earth in a circle as indicated on p. 8. The intersection of these three surfaces locates the points P and P'.

To obtain a rough estimate of some of the required accuracies, consider the orthogonal case represented by Eqs. (1) and (2), (i.e., where the relative velocity vector \bar{V}_r is normal to \bar{P} and \bar{S}). Then

$$\Delta P(r) = \frac{r \Delta r}{S \sin \gamma} \quad (6)$$

$$\Delta P(\psi) = \frac{r \Delta \dot{r}}{V_r} \quad (7)$$

Combining Eqs. (5), (6), and (7) gives

$$\Delta P(r) \approx \frac{r}{S} \frac{c \Delta \tau}{2} \approx \frac{r^2}{S} \frac{\Delta \tau}{\tau} \quad (8)$$

$$\Delta P(\psi) \approx \frac{c \Delta f_d}{2f} \cdot \frac{r}{V_r} \quad (9)$$

where

f is the transmitted frequency

Δf_d is the error in doppler measurement

$\Delta \tau$ is the error in time delay measurement

To estimate the required equipment accuracy, assume that the satellite is at synchronous altitude in an equatorial plane and is rotating in a direction opposite to that of the earth, i.e., retrograde orbit. Then, in twelve hours the satellite will make one revolution with respect to the earth so that

$$V_r = \frac{2\pi a R_e}{t}$$

where

$$t = 12 \text{ hr}$$

$$a = 6.61 \text{ earth radii}$$

$$R_e = 3,437.5 \text{ n mi}$$

$$c = .572 \times 10^9 \text{ n mi/hr}$$

Then

$$\Delta P(r) \text{ (n mi)} \sim 2 \times 10^4 \frac{\Delta \tau}{\tau}$$

$$\Delta P(\psi) \text{ (n mi)} \sim 6 \times 10^8 \frac{\Delta f_d}{f}$$

This means that frequency must be stable to about one part in 10^9 during the time of measurement to obtain an accuracy of one n mi

in the ψ direction. Considerably less stability is required to make the range measurements to the same accuracy (because the time period can be much less). It is interesting to note that some doppler radars that must reject very high clutter ratios require frequency stabilities of one part in 10^{10} or better for short time periods.⁽⁷⁾ These conditions can be met by using a master crystal oscillator, multiplier chain and power amplifier (MOPA); in fact, they are met in some airborne radars.

To make position measurements to a given accuracy requires, in addition to stability, a certain bandwidth B for the range measurement, and a measurement time T for the velocity or angle measurement. These quantities B and T determine the resolution in range and velocity that can be achieved. Since it is possible to obtain better accuracy than resolution if the signal-to-noise ratio is sufficiently large, there is a tradeoff between the quantities B , T and e/n , the received signal-to-noise ratio. When higher derivatives of range and velocity are not significant, the relations are

$$\Delta\tau = \frac{k_1}{B\sqrt{e/n}} = \frac{K}{B} \quad (10)$$

$$\Delta f_d = \frac{k_1}{T\sqrt{e/n}} = \frac{K}{T} \quad (11)$$

where

B = signal bandwidth

T = time taken to make the measurement and the reciprocal of the effective noise bandwidth

e/n = the received signal energy-to-noise ratio

k_1 = a constant dependent on energy distribution
($\sim \sqrt{3/\pi}$ for a uniform distribution)

These relations are derived in many places. (8-12)

In the above relations the assumption is made that the quantities are not changing during the measurements. That is, the satellite must not move appreciably during the time of measurement

$$V_r T < \Delta P(\psi) \quad (12)$$

This condition is not necessary, that is, $V_r T$ could be larger than $\Delta P(\psi)$; however, if so, it is necessary to measure acceleration as well as velocity. The data processing would be somewhat more complicated, and other relations would be required relating energy to accuracy. (12)

Assuming that the inequality of Eq. (12) exists, the angle measurement places a limit on the time that can be taken to make the measurement, and therefore on the measurement accuracy. Combining Eqs. (9) and (11) yields

$$\Delta P(\psi) \approx \frac{r \lambda K}{2 V_r T} \quad (13)$$

subject to the condition $\Delta P(\psi) > V_r T$. Thus

$$\left(\Delta P(\psi) \right)^2 > \frac{r \lambda K}{2} \quad (14)$$

Setting $K = 1$ is equivalent to equating accuracy to resolution. However, for good detection e/n must be between 10 and 100; therefore, reasonable values of K are from about .15 to .055.

Next, consider the one-way energy required to make the measurements.

The transmitted energy is given by the peak power \hat{P} times its duration; thus

$$E = \hat{P}T \quad (15)$$

assuming the transmission is continuous over the period T .

The received signal peak power amplitude is

$$s = \frac{\hat{P}GA}{4\pi r^2}$$

and the effective noise power in the receiver is

$$n = \frac{k T_{\text{eff}}}{T}$$

where T is the total integration time, k is Boltzmann's constant and T_{eff} is the effective temperature including system losses. Thus, if a measurement is made over a time period T , then

$$\hat{P}T = E = \frac{4\pi e/n r^2 k T_{\text{eff}}}{GA} \quad (16)$$

If the satellite has sufficient gain so that all of the earth is illuminated, and if the receiver is a dipole, then

$$G \sim \frac{4\pi r^2}{\pi R_e^2} = 4a^2$$

$$A \sim \frac{\lambda^2}{4\pi}$$

and the energy required for a one-way transmission is

$$E = \frac{4\pi^2 e/n r^2 k T_{eff}}{\lambda^2 a^2} \quad (17)$$

If r is replaced by a R_e , where R_e is the radius of the earth, then

$$E = \frac{4\pi^2 e/n k T_{eff}}{\lambda^2} \cdot R_e^2 \quad (18)$$

Assuming no limit to the size of the antenna in the satellite, the power required in the satellite is independent of the satellite altitude, to a good approximation ($a \gg 1$).

Actually, if a sufficiently large antenna were used, the gain could be adjusted to be lower directly under the satellite (where the range is least), and to increase toward the earth limb (where the range is greater). This would produce a small additional saving in power. The major gain would accrue to low-altitude satellites.

The choice of noise temperature for the receiver on the ground considerably affects the energy required in the satellite. There appear to be roughly three choices.

Type	Maser	Tunnel Diode	Crystal ^b
Noise Temp ^a (Kelvin)	60°	600°	3000°
Cost	\$25,000	\$1,000	\$100
Life	2000 hrs	Very Long	Long

^aThese noise temperatures include all receiver losses.

^bThese data were obtained from Ned Feldman of The RAND Corporation.

Consider, for instance, C- and L-band with $e/n = 100$, which makes $K = .055$ and set noise temperature = 600°.

$$\lambda = .2 \text{ ft (C-band)}$$

$$\lambda = 1 \text{ ft (L-band)}$$

$$B = \frac{4.4 \times 10^4}{\Delta P(r) (n \text{ mi})} \text{ (cps)}$$

$$B = \frac{4.4 \times 10^4}{\Delta P(r) (n \text{ mi})}$$

$$\Delta P(\psi) > .14 \text{ n mi}$$

$$\Delta P(\psi) > .32 \text{ n mi}$$

$$T = \frac{.019}{\Delta P(\psi) (n \text{ mi})} \text{ sec}$$

$$T = \frac{.095}{\Delta P(\psi) (n \text{ mi})} \text{ sec}$$

$$E = .45 \text{ joule}$$

$$E = .018 \text{ joule}$$

If the value of K is taken as unity, no interpolation is required, i.e., the processing is quite simple. Note that in this case

$$\lambda = .2 \text{ ft (C-band)}$$

$$\lambda = 1 \text{ ft (L-band)}$$

$$B = \frac{8 \times 10^5}{\Delta P(r) \text{ (n mi)}}$$

$$B = \frac{8 \times 10^5}{\Delta P(r) \text{ (n mi)}}$$

$$\Delta P(\psi) > .6 \text{ n mi}$$

$$\Delta P(\psi) > 1.35 \text{ n mi}$$

$$T = \frac{.344}{\Delta P(\psi) \text{ (n mi)}} \text{ sec}$$

$$T = \frac{1.72}{\Delta P(\psi) \text{ (n mi)}} \text{ sec}$$

$$E = .45 \text{ joule} \\ (e/n = 100)$$

$$E = .018 \text{ joule} \\ (e/n = 100)$$

Many waveforms can be used to make the measurements discussed above. Both range and velocity measurements can be made simultaneously using the same transmitted energy. This can be done using a transmission that lasts for the period T and covers a frequency band B. Energy can be distributed in the time frequency domain to give many different types of so-called "ambiguity diagrams."⁽¹³⁾ If, for instance, the distribution is pseudorandom, the ambiguity function can be "thumb tack" in shape, which means that each pair of points P and P' will have a unique matched filter output. While this approach is most general and economical, the processing may be complicated.

Simplifications in processing may be achieved by making the range and velocity measurements with two separate transmissions or by using more ambiguous functions. One waveform that is interesting because of its frequent use in radar is a series of pulses either evenly or unevenly spaced. The series must last for at least a period T to make

the velocity measurement. If few pulses are used, there will be ambiguities in the measurement of the angle ψ . If, for instance, a single pulse were used at the beginning and end of a period T and if $K = 1$, then each interval $\Delta P(\psi)$ would be ambiguous because the frequency would change by one cycle in the time T for each change in position $\Delta P(\psi)$.

If a train of uniformly spaced pulses were used, there would be range ambiguities. If the range between pulses is r_b , the ambiguity in the ψ direction r_u is given by

$$r_u = \frac{c}{4} \frac{r\lambda}{V r_b} \quad (19)$$

If r_b is to be equal to the radius of the earth so that there will be no ambiguity in measurement of range, then, using the same values as before

$$r_u (\text{n mi}) \sim 13.2 \lambda (\text{ft})$$

This would probably not be satisfactory operationally. If, instead, the ambiguities were divided between the r and ψ coordinates, then

$$r_u (\text{n mi}) \ r_b (\text{n mi}) \sim \frac{cr\lambda}{4v} \sim 4.55 \times 10^4 \lambda (\text{ft})$$

and if the ambiguities were shared equally in the r and ψ directions, then

$$r_u = r_b \sim 213 \text{ n mi } \sqrt{\lambda (\text{ft})}$$

which might be satisfactory for some applications. At the expense of further complexity, the spacing between pulses could be varied; this then removes the ambiguity. All the techniques for measuring range and velocity that have been discussed and applied to radar are of course applicable here, and since there is a wide choice, the selection of waveform would require further analysis.

It has been shown that a navigation system can be built which can measure position on the earth to an accuracy of one n mi or less by making a two-way range and doppler measurement between a satellite having a velocity relative to a point on the ground. The transmitter and receiver can be at either end, and a transponder is required at the other end. The instantaneous satellite vector velocity must be known to an angular accuracy

$$\Delta\psi < \frac{\Delta P(\psi)}{r}$$

and the clock must be accurate over a period T to better than one part in 10^9 for synchronous altitude. When these conditions are met, the position measurement can be made in less than a second for reasonable parameters. If the transmissions are initiated and received on the ground, say, from a master tracking station, then both the satellite and the navigator require a minimum of equipment and power. Basically each requires a transponder.

A satellite communication system could be considered as a potential way of determining location, since this same equipment could be used for communication. From this point of view, the data rate used

for navigation is one bit per period T , where the data are encoded using a bandwidth B . Hence, it is similar to communication over an effective bandwidth $1/T$. Since T is about .1 to 1 sec, the data rate is quite low even if position were updated every T sec. Actually, position would probably be required only once every 5 to 10 min, so that the energy used for position determining would be quite low compared to that required for most forms of communication.

This discussion has been concerned with determining the location of a fixed point on the earth when its altitude is known. If the altitude is not known or if the ground point is moving, errors in location are introduced.

The remainder of this Memorandum is concerned with a computational method for determining the position of a vehicle which is moving with respect to a rotating earth by obtaining several measurements of range and doppler data.

V. FIX DETERMINATION PROCESS

This section contains a description of an iterative computational technique involving differential corrections. It should be familiar to those acquainted with orbit determination processes.

An initial rough estimate of the user vehicle's position and velocity is required to start this process. (Experiments so far indicate that these estimates can be very rough indeed. In fact, it is quite likely that the user vehicle would not have to supply this information.) The basic idea of the scheme is to compare parameters obtained from the assumed position with those obtained from the measurements. Corrections to the assumed position are then computed in a systematic and orderly way so that the difference between assumed and measured values is reduced to zero.

To describe this process, consider a vehicle to be located on the surface of the earth at point P_1 and a satellite at point S_1 in space at time t_1 , as shown in Fig. 1. The coordinates of P_1 are defined by geographic longitude λ_{1P} and latitude φ_{1P} . The coordinates of S_1 are defined in a similar way using λ_{1S} , φ_{1S} for the subsatellite point. Assume arbitrarily that the vehicle's trajectory may be described by a rhumb-line course (defined as one where the course line makes the same oblique angle A with all meridians) and that the speed V is constant. Now postulate the following:

- o The orbit of satellite S is precisely known, i.e., λ_{1S} , φ_{1S} at time t_1 ; λ_{2S} , φ_{2S} at time t_2 ; etc., as well as its altitude h_{1S} , h_{2S} , etc.

- o Range r_1 between P_1 and S_1 and its rate of change \dot{r}_1 can be measured precisely by some means yet to be described; similarly, r_2 and \dot{r}_2 at time t_2 , etc.

The problem is how to find the four unknowns (in this restricted case) λ_{1P} , φ_{1P} , A , and V , given the four measured quantities r_1 , \dot{r}_1 , r_2 , and \dot{r}_2 . The satellite positions S_1 and S_2 are known quantities. The assumption is made that the trajectory between points P_1 and P_2 can be described by the equations relating to a constant speed rhumb-line course. Before proceeding with details and derivations perhaps it is better to outline briefly the procedure for obtaining the Δr and $\Delta \dot{r}$ residuals:

- o Estimate the vectors \bar{P}_1 and $\dot{\bar{P}}_1$, i.e., λ_{1P} , φ_{1P} , A , and V .
- o Using the rhumb-line equations, calculate \bar{P}_2 and $\dot{\bar{P}}_2$.
- o Using the above values derived from estimates, compute r_1 , r_2 , \dot{r}_1 , \dot{r}_2 .
- o Form the residuals $\Delta r_1 = r_{10} - r_{1c}$, $\Delta \dot{r}_1 = \dot{r}_{10} - \dot{r}_{1c}$, and $\Delta r_2 = r_{20} - r_{2c}$, $\Delta \dot{r}_2 = \dot{r}_{20} - \dot{r}_{2c}$ where the 0's refer to observed or measured quantities and the c's refer to computed values.

Using a Taylor series expansion in four dimensions and ignoring second and higher order terms we may write

$$\Delta r_1 = \frac{\partial r_1}{\partial \varphi_1} \Delta \varphi_1 + \frac{\partial r_1}{\partial \lambda_1} \Delta \lambda_1 + \frac{\partial r_1}{\partial A} \Delta A + \frac{\partial r_1}{\partial V} \Delta V$$

$$\Delta \dot{r}_1 = \frac{\partial \dot{r}_1}{\partial \varphi_1} \Delta \varphi_1 + \frac{\partial \dot{r}_1}{\partial \lambda_1} \Delta \lambda_1 + \frac{\partial \dot{r}_1}{\partial A} \Delta A + \frac{\partial \dot{r}_1}{\partial V} \Delta V$$
(20)

and similarly for Δr_2 , $\Delta \dot{r}_2$. Thus, two sets of observations provide

four residuals in Δr and $\Delta \dot{r}$, thereby allowing a solution for the corrections to the four elements, φ_1 , λ_1 , A , and V .

Now consider the more general case where (1) six unknowns are involved including altitude h and its rate of change \dot{h} , and (2) measurement errors are involved in r and \dot{r} values. The n sets of observations will provide $2n$ residuals in r and \dot{r} , thereby allowing a solution for the corrections to the six initial estimates of λ_1 , φ_1 , A , V , h , and \dot{h} . More than $n = 3$ sets will allow a more accurate fit by the method of least squares. Each set corresponds to a particular time and the partial derivative coefficients must be evaluated for this time. The matrix M , which must be inverted for determining the corrections, is then

$$\begin{bmatrix} \Delta r_1 \\ \Delta \dot{r}_1 \\ \Delta r_2 \\ \Delta \dot{r}_2 \\ . \\ . \\ . \\ \Delta \dot{r}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{2n1} & a_{2n2} & a_{2n3} & a_{2n4} & a_{2n5} & a_{2n6} \end{bmatrix} \begin{bmatrix} \Delta \varphi_1 \\ \Delta \lambda_1 \\ \Delta A \\ \Delta V \\ \Delta h \\ \Delta \dot{h} \end{bmatrix} \quad (21)$$

where a_{11} , a_{12} , ..., and a_{21} , a_{22} , ..., are the partial derivative coefficients evaluated at t_1 ; the next two rows are evaluated at t_2 and so on. Appendices B and C contain details regarding the

mathematical relationships necessary to this process. When n sets of these residuals are available they form a $2n \times 1$ residual vector Δa .

$$\Delta a = \begin{bmatrix} \Delta r_1 \\ \Delta \dot{r}_1 \\ \Delta r_2 \\ \Delta \dot{r}_2 \\ . \\ . \\ . \\ \Delta r_n \\ \Delta \dot{r}_n \end{bmatrix}$$

The least-squares solution for the errors in the estimates may be written in a shorthand vector matrix form

$$\Delta p = \left[M^T M \right]^{-1} M^T \Delta a \quad (22)$$

where Δp is the desired 6×1 error vector, and M is the partial derivative coefficient matrix of Eq. (21). This equation may also be written as

$$\Delta p = \begin{bmatrix} \Delta \lambda_1 \\ \Delta \varphi_1 \\ \Delta A \\ \Delta V \\ \Delta h \\ \Delta \dot{h} \end{bmatrix} = \begin{bmatrix} M^T M \end{bmatrix}^{-1} M^T \begin{bmatrix} \Delta r_1 \\ \Delta \dot{r}_1 \\ \Delta r_2 \\ \Delta \dot{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta r_n \\ \Delta \dot{r}_n \end{bmatrix}$$

(6 x 2 n matrix)

The elements of the Δp error matrix represent corrections which are to be added to the previous values of λ_1 , φ_1 , A , V , h , and \dot{h} . The process is then repeated until the errors in the elements of the residual vector Δa are sufficiently small.

VI. SIMULATION RESULTS

A simplified model of the process described above was developed and simulated on a digital computer for a variety of conditions, e.g., satellite-vehicle geometry, speeds, time intervals, measurement errors, etc., in order to test for sensitivity to error and convergence.

Rather than simulating the general case of a six-element fix with least squares fitting and $2n$ measurements over n time intervals, we chose to consider a case where only latitude, longitude, speed, and course are unknown, i.e., r_1, \dot{r}_1 at time t_1 and r_2, \dot{r}_2 at time t_2 .

Because of the large number of parameters needed to define the time-varying paths of both satellite and vehicle, it was convenient to hold certain of these parameters constant in order to study the effects of varying others. Accordingly, the following initial conditions were adopted at $t = 0$ (unless otherwise noted on the figures):

- o Vehicle on the equator at 0° longitude and latitude
- o Vehicle heading 45° from north
- o Satellite on the equator 15° west from the vehicle
- o Circular satellite orbits

With this arrangement it was then convenient to vary

- o Satellite altitude and inclination angle
- o Vehicle speed
- o Times t_1 and t_2 at which measurements occurred (thereby defining relative position geometry and time interval between measurements)
- o Effective measurement errors (Δr and $\Delta \dot{r}$)

One of the first points of interest occurs in the fix determination capability of the process under adverse geometrical conditions which cause computational difficulties. As might be expected, these difficulties will appear in the form of "ill-conditioned" matrices such as occur in linear algebraic equations of the form

$$Ax = y$$

where x and y are vectors and A is a square matrix. If the determinant of A , though not zero, is very small in absolute value, the numerically large elements in A^{-1} will amplify small numerical errors occurring in the calculation to the point that they overwhelm the significant elements. Figure 2, taken from Ref. 14, illustrates a typical instability obtained using an iterative process involving an ill-conditioned matrix inversion. In this navigation process, indeterminacy (or instability) is approached primarily as a function of the relative satellite-vehicle motion between measurements. Therefore, greater computational difficulties might be expected at (1) higher satellite altitudes, and (2) shorter time intervals between measurements. The approach taken was to perform a number of idealized cases (i.e., error-free measurements) in order to check the program answers against a known result, i.e., zero navigation error. The answers are considered to be "zero" if the numerical values are one or, hopefully, two orders of magnitude smaller than a desired maximum of one n mi. Once the computational error has been isolated, and if it is sufficiently small, a given quantity is perturbed in order to establish an error sensitivity. Of course, if a computational "zero" cannot be established for an idealized

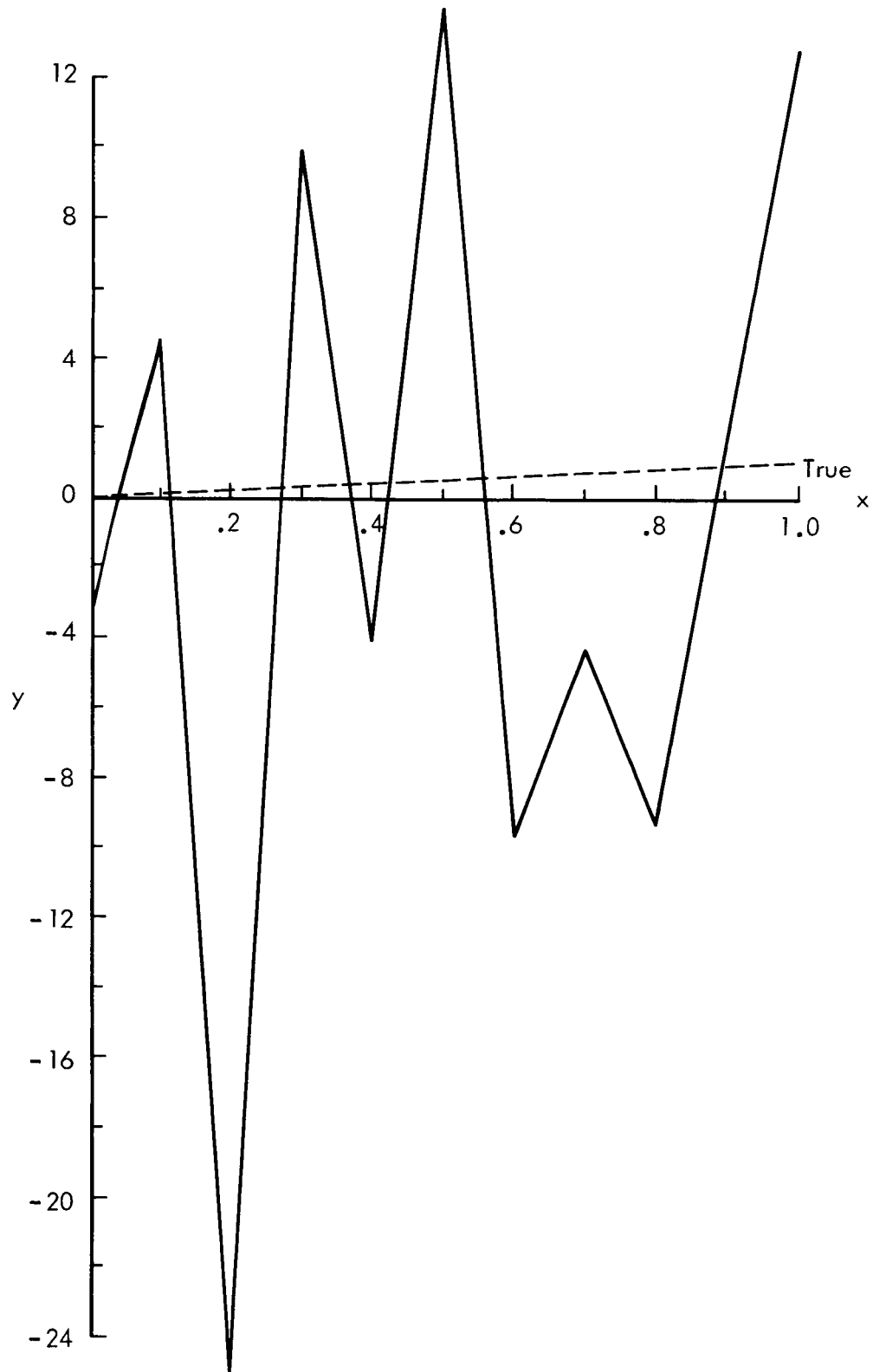


Fig.2—Solution by matrix inversion

case because of instability, the error sensitivity determination is not too meaningful. Figure 3 shows computer results for two idealized cases (i.e., error-free measurements) with a satellite in a circular orbit at 12,000 n mi altitude. Note that with both cases (i.e., five and ten minute time intervals between measurements) the process tends to converge quickly from the initial position estimate. However, the five minute case, instead of converging to zero as it should, approaches the three mile error level and oscillates about this value, failing to improve. The navigation errors represented in these curves are due entirely to amplification of small numerical errors by the inverse of an ill-conditioned matrix. Therefore, the numerical answers are dependent on the particular machine accuracy and numerical processes used. There are several processes for inverting matrices with varying degrees of accuracy and there are differential correction techniques for improving a given inversion. Moreover, there are other computational techniques for solving a system of linear equations by successive approximation and/or dynamic programming.

Not being satisfied with the results indicated by Fig. 3 in establishing a zero, we supplemented the program with a numerical technique as described in Refs. 14 and 15, and were successful in reducing the numerical errors by one and, in some cases, two orders of magnitude. However, the basic root of the difficulty is that the system error sensitivity is very great at the higher satellite altitudes. Subsequent results will show that this sensitivity, acting upon errors from other sources, will overwhelm the numerical errors so as to make these cases rather uninteresting. For example, Fig. 4 shows results

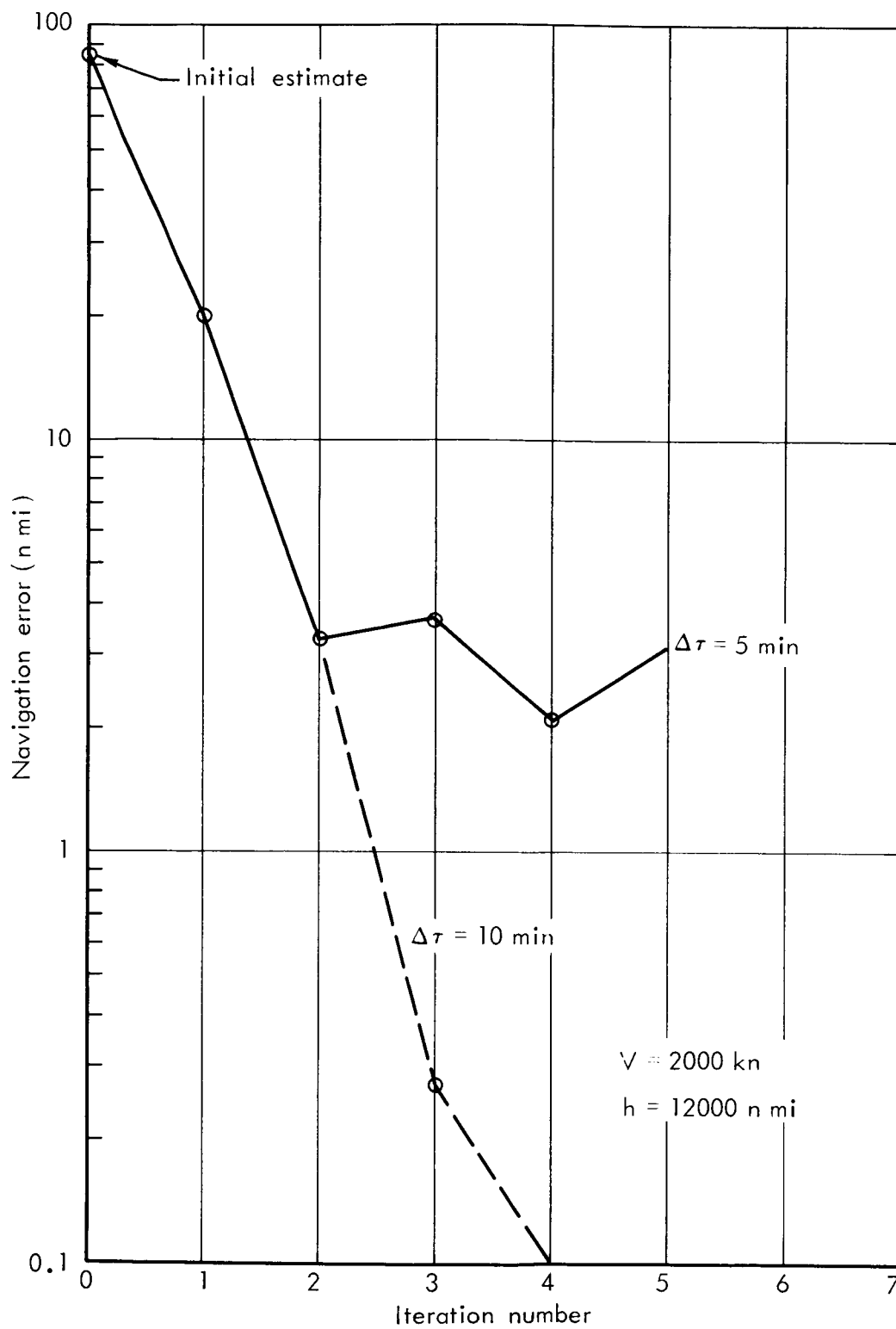


Fig.3—Effects of time interval on convergence
(error free measurements)

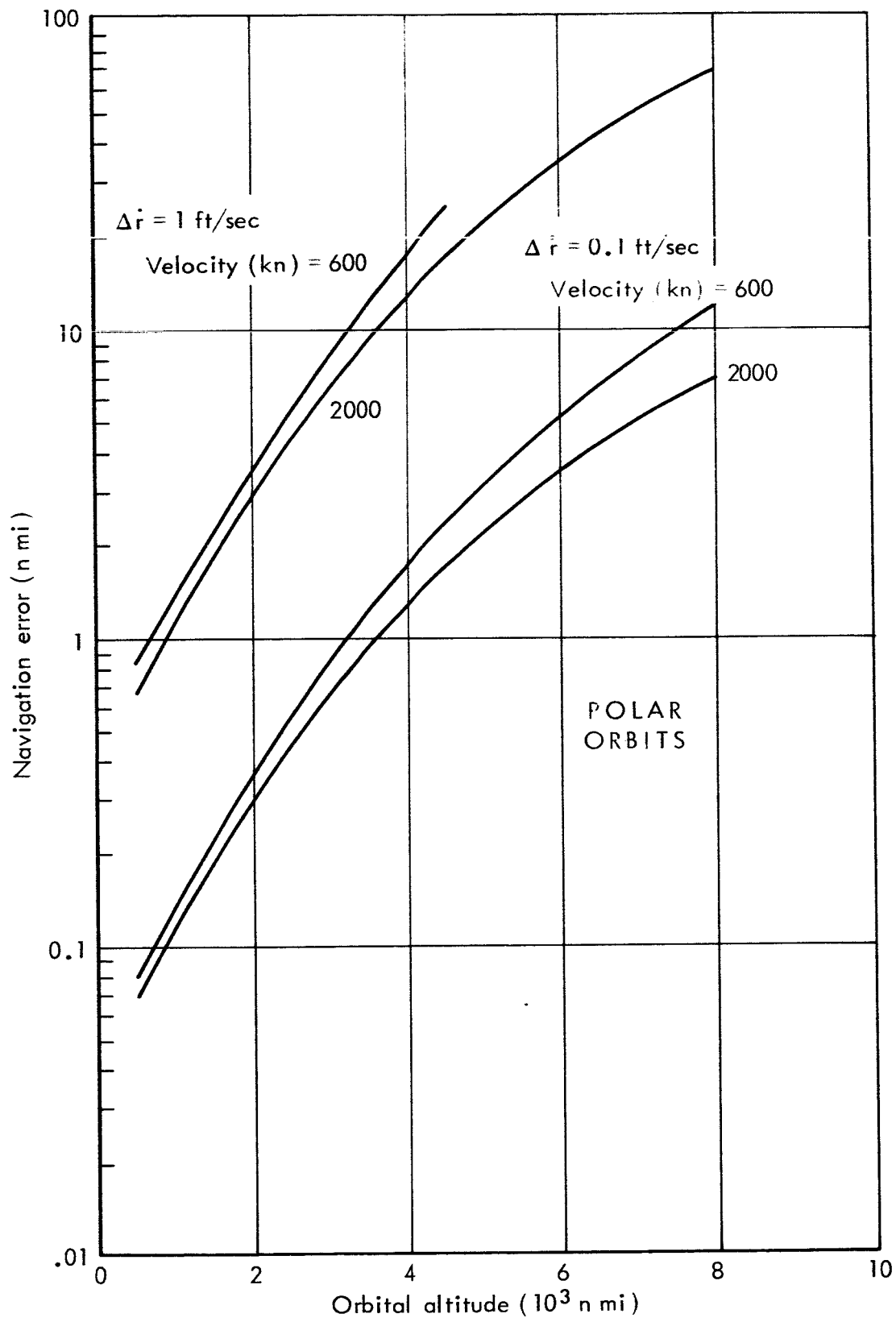


Fig.4—Navigation errors due to $\Delta \dot{r}$ measurement error

for a number of cases where the time interval between the two sets of measurements was 10 min. Two sets of error curves are shown, one for a 1.0 ft/sec range-rate $\Delta \dot{r}$ measurement perturbation, the other for a perturbation of 0.1 ft/sec. Note that there is a linear relationship, in that the 1.0 ft/sec error curves are displaced vertically one order of magnitude above the 0.1 ft/sec curves, the slopes are steep, and the resultant error quickly becomes unacceptable at the higher altitudes. Figure 5 shows the results obtained when the time interval between measurements was varied for a constant altitude case ($h = 2000$ n mi). The results for direct and retrograde equatorial orbits are compared to illustrate the influence of the relative velocity on the resultant errors. (See Fig. 6.)

Three different major error sources may contribute to an effective $\Delta \dot{r}$: (1) electronic measurement errors due to frequency instability or propagation effects, (2) uncertainties in satellite velocity, and (3) vehicle accelerations during the measurement time interval. This last effect may be regarded as equivalent to a range-rate measurement error, since the navigation process is based on the assumption that a constant speed rhumb-line course prevails during a time interval which is to be small. Accordingly, the resultant solution to the navigation problem involves a constant average velocity between points. If acceleration exists, the instantaneous vehicle velocities at times t_1 and t_2 will differ from this average value, effectively contributing to a $\Delta \dot{r}$ error. Again this points to the desirability of many measurements at short time intervals. It follows that the data processing must be compatible with the computational accuracy imposed by the short time intervals.

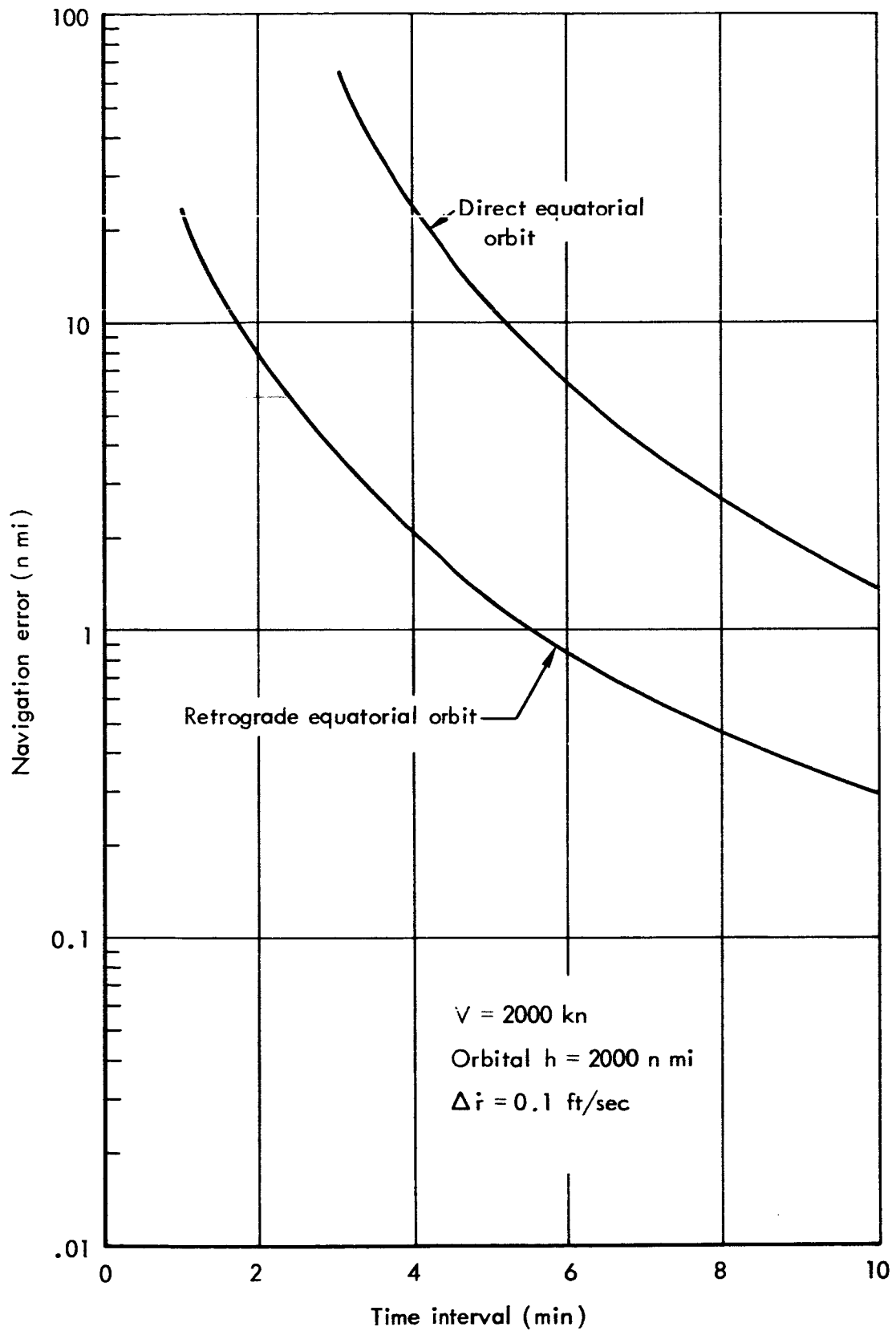


Fig.5—Effects of measurement time interval on navigation error

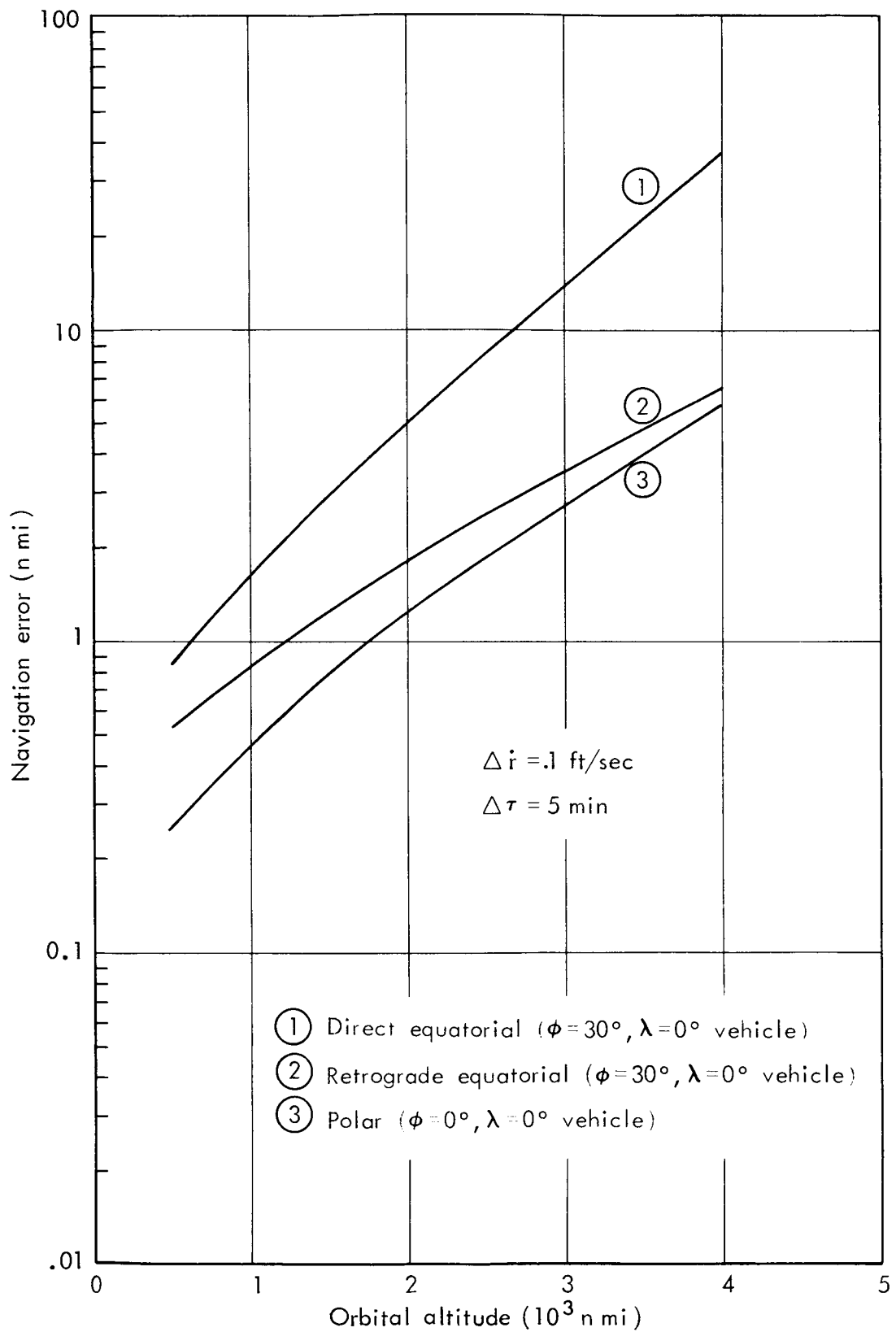


Fig. 6—Navigation error due to $\Delta \dot{r}$ measurement error

VII. CONCLUSIONS

The results from the previous section, although by no means complete, are sufficient to draw certain conclusions regarding the feasibility and practicality of this concept for aircraft navigation. To review briefly, the basic idea is to determine the aircraft's position (accurately) from multiple measurements, using a single satellite, without requiring the aircraft to provide precise velocity information. That is, the aircraft velocity information necessary to compute the fix (since a time interval between measurements is involved) will be derived from data processing by a ground station. The problem is difficult because it is necessary to separate out computationally the contribution of the vehicle's velocity to the change in measurements. This contribution is likely to be small relative to that caused by the satellite and the earth's rotation. Moreover, the resultant navigation error is quite sensitive to errors in the determination of the vehicle's velocity.

The simulation results show that the data processing technique is rapidly convergent, e.g., from an initial estimate of an 84-mi error to zero in about four iterations. The navigation process is quite sensitive to effective errors in measured doppler shift, particularly at the higher satellite orbits where the relative velocity decreases. Orbital altitudes greater than about 4000 mi show unsatisfactory aircraft navigation accuracies for the estimated range of values for $\Delta \dot{r}$ error and for practical time intervals between measurements. However, for surface vessels where the time interval could be stretched out, it is likely that sufficient accuracy may be achieved

using the higher altitudes. Low-altitude orbits are most favorable for accuracy because of the higher relative velocities. The most unfavorable geometry occurs when the satellite is directly over the vehicle or when the satellite's velocity is roughly in the same plane as the great circle connecting the subsatellite point and the vehicle.

The work done so far has been preliminary in nature and has not included the generalized model of a "six-dimensional" fix with many measurements over a time interval using a least-squares fit. Certainly this would have to be done before detailed conclusions can be drawn regarding (1) the effects of vehicle acceleration and its contribution to effective $\Delta \dot{r}$ error, (2) lower and upper practical limits on the fix time interval and (3) effects of random error on navigation accuracy. The results of this investigation indicate that further study efforts are warranted if low-altitude communication satellites become operational so that these navigational techniques might be employed as a bonus benefit from the basic communications satellite system.

Appendix A

SIMPLIFIED THREE-DIMENSIONAL ERROR ANALYSIS

In Section III only the positional geometry in the plane formed by the vectors \bar{P} , \bar{S} , and \bar{r} was considered. It is worthwhile to examine the general three-dimensional case where the relative velocity vector \bar{V}_r (or $\dot{\bar{r}}$) is not necessarily coplanar. The basic equations defining the navigation fix problem may be expressed by the simple vector equation

$$\bar{P} = \bar{S} + \bar{r} \quad (1a)$$

There are three unknowns to be determined: P_x, P_y, P_z . For this example assume that the vectors \bar{S} , $\dot{\bar{S}}$, and $\dot{\bar{P}}$ are precisely known and hence the relative velocity $\dot{\bar{r}}$ is defined by

$$\dot{\bar{r}} = \dot{\bar{P}} - \dot{\bar{S}} \quad (2a)$$

where all velocities are referred to an inertial nonrotating frame. Two quantities are measured: r and \dot{r} . For simplicity, let us take

$$|\bar{P}| = P = 1 \text{ (earth radii)}$$

The solution of the following three equations will yield the three components of \bar{P}

$$\begin{aligned} \bar{P} \cdot \bar{P} &= 1 \\ \bar{r} \cdot \bar{V}_r &= \dot{r}r \\ \bar{r} \cdot \dot{\bar{r}} &= r^2 \end{aligned} \quad (3a)$$

Of particular interest is the relationship between the position error $\Delta\bar{P}$ and the errors in measured quantities Δr and $\Delta\dot{r}$. Differentiating the above equations and expressing the results in scalar form

$$P_x \Delta P_x + P_y \Delta P_y + P_z \Delta P_z = 0 \quad (4a)$$

$$V_{rx} \Delta r_x + V_{ry} \Delta r_y + V_{rz} \Delta r_z = \dot{r} \Delta r + r \Delta \dot{r} \quad (4a)$$

$$r_x \Delta r_x + r_y \Delta r_y + r_z \Delta r_z = r \Delta r$$

where $r_x = P_x - S_x$, $\Delta r_x = \Delta P_x$, and similarly for the y and z components. At this point, a convenient x, y, z coordinate system can be arbitrarily chosen without loss of generality. For the moment, postulate that there is an x, y, z nonrotating frame such that \bar{P} is along the x-axis. That is

$$P_x = P = 1$$

$$P_y = P_z = 0$$

Since the latitude and longitude errors in $\Delta \bar{P}$ are of primary concern, errors in altitude will be ignored so that

$$\Delta P_x = 0$$

Now, the resultant solutions to Eqs. (4a) may be expressed as

$$\Delta P_y = \frac{(r_z \dot{r} - V_{rz} r) \Delta r + r_z r \Delta \dot{r}}{(V_{ry} r_z - r_y V_{rz})} \quad (5a)$$

$$\Delta P_z = \frac{(r V_{ry} - r_y \dot{r}) \Delta r - r_y r \Delta \dot{r}}{(V_{ry} r_z - r_y V_{rz})} \quad (6a)$$

It is interesting to note that the denominator in the above expressions may be written as

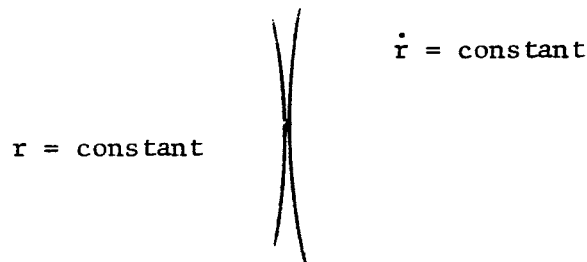
$$\bar{P} \cdot (\bar{V}_r \times \bar{r})$$

which is identical to

$$\bar{V}_r \cdot (\bar{r} \times \bar{P})$$

The vector $(\bar{r} \times \bar{P})$ is normal to the plane containing the position vectors \bar{r} , \bar{P} and \bar{S} . If \bar{V}_r should be in this plane, the above dot product and hence the denominator in Eq. (6a) will be zero. Geometrically, this may be interpreted as the situation where the two lines of position (LOPs) are parallel. This will occur when the two arcs (i.e., $r = \text{constant}$ sphere and $\dot{r} = \text{constant}$ cone) on the earth's surface are tangent.

Referring to Fig. 1, note that if \bar{V}_r is in the same plane as \bar{P} and \bar{S} , the intersection on the earth's surface will not be as shown on p. 8 but as shown in the sketch below.



Therefore, the most favorable geometry for minimum error occurs when the relative velocity \bar{V}_r is normal to the \bar{r} , \bar{S} , \bar{P} plane.

In choosing a coordinate system, it was only stipulated that \bar{P} should be along the x-axis. The results may be simplified further by choosing the coordinate system such that the x, y plane contains \bar{r} , \bar{S} , and \bar{P} so that

$$P_x = P = 1$$

$$P_y = P_z = r_z = S_z = 0$$

The results of Eqs. (4a) then simplify to

$$\Delta P_y = \frac{r\Delta r}{r_y} = \frac{r\Delta r}{S \sin \gamma} \quad (7a)$$

$$\Delta P_z = - \frac{\left[\left(v_{ry} \frac{r}{r_y} - \dot{r} \right) \Delta r - r\Delta \dot{r} \right]}{V_{rz}} \quad (8a)$$

$$\Delta P = \left[\Delta P_y^2 + \Delta P_z^2 \right]^{\frac{1}{2}} \quad (9a)$$

Note that the expression for ΔP_y is the same as given by Eq. (2) and that a singularity exists when the out-of-plane component V_{rz} is zero. Moreover, if \bar{V}_r is orthogonal to the $\bar{r}, \bar{S}, \bar{P}$ (x, y) plane so that $V_{rx} = V_{ry} = 0$, then the expression for ΔP_z reduces to

$$\Delta P_z = \frac{r\Delta \dot{r}}{V_r} \quad (10a)$$

The above result is comparable to Eq. (1). In the work that follows, the satellite velocity $\dot{\bar{S}}$ is expressed in terms of inertial coordinates, i.e., nonrotating frame of reference. The vehicle velocity $\dot{\bar{P}}$ is then referred to this same frame and must necessarily contain terms involving the earth's rotation.

As noted in Section III, it is important to recall that \bar{V}_r is a relative velocity expressed in an earth-fixed rotating coordinate system so that

$$\bar{V}_r = \dot{\bar{r}} - \bar{\omega}_e \times \bar{r}$$

For low altitude, large absolute values of \bar{V}_r may be expected due principally to the satellite motion. On the other hand, for a circular, equatorial, synchronous satellite, the relative motion expressed by \bar{V}_r is due only to the vehicle's motion and is zero for a fixed point on the earth. However, the relative motion as defined by $\dot{\bar{r}}$ in an inertial system would not be zero for the above mentioned case or in any practical case.

Appendix B

FORMULAS APPLYING TO FIX DETERMINATION PROCESS

The purpose of this Appendix is to set down certain mathematical relationships which are necessary to the numerical process described in Section V. Returning to the example for $n=2$ sets of observations, the four quantities r_{o1} , \dot{r}_{o1} , r_{o2} , and \dot{r}_{o2} are measured and it is necessary to compute the values of r_{c1} , \dot{r}_{c1} , r_{c2} , and \dot{r}_{c2} using estimates for φ , λ , A , and V . First, we must have the basic equations describing the trajectory of the vehicle between observation times t_1 and t_2 . These equations may be established by taking an element of arc as

$$dS = P \left[\cos^2 \varphi d\lambda^2 + d\varphi^2 \right]^{\frac{1}{2}} \quad (1b)$$

and

$$\tan A = \cos \varphi \frac{d\lambda}{d\varphi} \quad (2b)$$

These two equations may be integrated as follows

$$\begin{aligned} V(t_2 - t_1) &= \int_{t_1}^{t_2} dS = \int_{\varphi_1}^{\varphi_2} P \left[\cos^2 \varphi \left(\frac{d\lambda}{d\varphi} \right)^2 + 1 \right]^{\frac{1}{2}} d\varphi \\ &= P(\varphi_2 - \varphi_1) \sec A \end{aligned}$$

Hence

$$\begin{aligned} \varphi_2 &= \varphi_1 + \frac{V(t_2 - t_1)}{P} \cos A \\ \lambda_2 - \lambda_1 &= \tan A \int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\cos \varphi} \end{aligned} \quad (3b)$$

so that

$$\lambda_2 = \lambda_1 + \tan A \ln \left[\frac{\tan \varphi_2 + \sec \varphi_2}{\tan \varphi_1 + \sec \varphi_1} \right] \quad (4b)$$

Referring to Fig. 1, the values for range may be computed from the following vector expressions. The subscripts 1 and 2, indicating values at times t_1 and t_2 , are dropped for generality.

$$|\bar{r}| = |\bar{P} - \bar{S}|$$

In an inertial, equatorial, nonrotating reference frame, the components of \bar{P} are given by

$$\begin{aligned} P_x &= P \cos \varphi \cos \theta \\ P_y &= P \cos \varphi \sin \theta \\ P_z &= P \sin \varphi \end{aligned} \quad (5b)$$

where

$$\theta = \lambda + n_E t$$

n_E = earth's rotational rate

t = Greenwich sidereal time (i.e., taking the x-axis along the vernal equinox)

The components of \bar{S}_1, \bar{S}_2 may be calculated from the orbital parameters which are assumed to be precisely known. Appropriate expressions are set down here for convenience. (The reader is referred to any text on celestial mechanics, such as Ref. 16, for derivation.)

$$S_x = x_\omega p_x + y_\omega q_x$$

$$S_y = x_\omega p_y + y_\omega q_y$$

$$S_z = x_\omega p_z + y_\omega q_z$$

where

$$x_\omega = a(\cos E - e)$$

$$y_\omega = a \sqrt{1 - e^2} \sin E$$

$$p_x = \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i$$

$$p_y = \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i$$

$$p_z = \sin \omega \sin i$$

$$q_x = -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i$$

$$q_y = -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i$$

$$q_z = \cos \omega \sin i$$

a = semimajor axis of the orbit

E = eccentric anomaly given by the solution
of Kepler's equation $n(t - T) = E - e \sin E$

T = time of perigee passage

e = orbital eccentricity

i = inclination of the orbital plane

Ω = node angle

ω = argument of perigee

The values for range rate may be computed from

$$\dot{r} = \left[\dot{\bar{P}} - \dot{\bar{S}} \right] \cdot \frac{\bar{r}}{r} \quad (6b)$$

Again, subscripts are dropped for generality. Designating the velocity of the point P as \bar{V}_P , i.e., $\bar{V}_P = \dot{\bar{P}}$, and differentiating the expressions for \bar{P} and Eqs. (3b) yields the following

$$\dot{\varphi} = \frac{V}{P} \cos A$$

$$\dot{\theta} = \frac{V}{P} \sin A \sec \varphi + n_E$$

$$\begin{aligned} V_{Px} = & -V [\cos A \sin \varphi \cos \theta + \sin A \sin \theta] \\ & - P n_E \cos \varphi \sin \theta + \frac{\dot{h} P_x}{P} \end{aligned} \quad (7b)$$

$$\begin{aligned} V_{Py} = & -V [\cos A \sin \varphi \sin \theta - \sin A \cos \theta] \\ & + P n_E \cos \varphi \cos \theta + \frac{\dot{h} P_y}{P} \end{aligned}$$

$$V_{Pz} = V \cos A \cos \varphi + \frac{\dot{h} P_z}{P}$$

Designating the velocity of the satellite as \bar{V}_S , i.e., $\bar{V}_S = \dot{\bar{S}}$, the components referred to the same inertial frame are derived from

$$\dot{\bar{S}} = \bar{V}_S = -\frac{na^2}{r} \left[\sin E \bar{p} - \sqrt{1 - e^2} \cos E \bar{q} \right] \quad (8b)$$

where n = mean angular rate of the satellite (i.e., 2π divided by the period) and the components of \bar{p} and \bar{q} are given above.

At this point Eq. (6b) can be written in scalar form for computing range rate in terms of the scalar quantities listed above.

$$\dot{r} = \frac{(V_{Px} - V_{Sx}) r_x + (V_{Py} - V_{Sy}) r_y + (V_{Pz} - V_{Sz}) r_z}{[r_x^2 + r_y^2 + r_z^2]^{\frac{1}{2}}} \quad (9b)$$

Appendix C

EVALUATION OF PARTIAL DERIVATIVES

This appendix is concerned with the determination of the partial derivative coefficients of the matrix of Eq. (21), which is repeated here with the h and \dot{h} terms omitted.

$$\begin{bmatrix} \Delta r_1 \\ \Delta \dot{r}_1 \\ \Delta r_2 \\ \Delta \dot{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta r_n \\ \Delta \dot{r}_n \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial \phi_1} & \frac{\partial r_1}{\partial \lambda_1} & \frac{\partial r_1}{\partial A} & \frac{\partial r_1}{\partial V} \\ \frac{\partial \dot{r}_1}{\partial \phi_1} & \frac{\partial \dot{r}_1}{\partial \lambda_1} & \frac{\partial \dot{r}_1}{\partial A} & \frac{\partial \dot{r}_1}{\partial V} \\ \frac{\partial r_2}{\partial \phi_1} & \frac{\partial r_2}{\partial \lambda_1} & \frac{\partial r_2}{\partial A} & \frac{\partial r_2}{\partial V} \\ \frac{\partial \dot{r}_2}{\partial \phi_1} & \frac{\partial \dot{r}_2}{\partial \lambda_1} & \frac{\partial \dot{r}_2}{\partial A} & \frac{\partial \dot{r}_2}{\partial V} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial r_n}{\partial \phi_1} & \frac{\partial r_n}{\partial \lambda_1} & \frac{\partial r_n}{\partial A} & \frac{\partial r_n}{\partial V} \\ \frac{\partial \dot{r}_n}{\partial \phi_1} & \frac{\partial \dot{r}_n}{\partial \lambda_1} & \frac{\partial \dot{r}_n}{\partial A} & \frac{\partial \dot{r}_n}{\partial V} \end{bmatrix} \begin{bmatrix} \Delta \phi_1 \\ \Delta \lambda_1 \\ \Delta A \\ \Delta V \end{bmatrix}$$

Although the assumption of a rhumb-line course ($A = \text{constant}$ and $V = \text{constant}$) leads to range and range-rate expressions which appear relatively simple, it was surprising that evaluation of the partial derivatives was so tedious and the resultant expressions so inordinately complex.

Further reflection shows that a rhumb-line course over a spherical earth represents a rather complex motion in three-dimensional inertial space, as is indicated by reference to Eq. (7b).

Therefore, in an effort to simplify the differentiation in orderly fashion, the partial derivative expression will be broken down into parts which may be more readily evaluated.

The expression for range is

$$r = |\bar{r}| = |(\bar{P} - \bar{S})|$$

where \bar{P} and \bar{S} are the vehicle and satellite positions respectively. By partial differentiation with respect to the variables of concern, namely φ_1 , λ_1 , A and V

$$\frac{\partial r}{\partial q_i} = \frac{\partial(|\bar{r}|)}{\partial q_i} = \frac{\partial(|\bar{P} - \bar{S}|)}{\partial q_i}, \quad i = 1, 2, 3, 4$$

where

$$q_1 = \varphi_1$$

$$q_2 = \lambda_1$$

$$q_3 = A$$

and

$$q_4 = V$$

The range vector \bar{r} may be expressed in terms of its components, r_x , r_y , r_z so that

$$\frac{\partial \mathbf{r}}{\partial q_i} = \left(\frac{\partial \mathbf{r}}{\partial r_x} \frac{\partial \mathbf{r}}{\partial r_y} \frac{\partial \mathbf{r}}{\partial r_z} \right) \begin{bmatrix} \frac{\partial r_x}{\partial q_i} \\ \frac{\partial r_y}{\partial q_i} \\ \frac{\partial r_z}{\partial q_i} \end{bmatrix}, \quad i = 1, 2, 3, 4$$

The q_i represent the four parameters φ , λ , A , V at time t_1 . (Note that for the sake of generality, subscripts which indicate a particular time have been dropped except where they are needed to clarify the derivation of the partial derivatives.)

The $\bar{\mathbf{r}}$ components, r_x , r_y , and r_z , in turn are each functions of the x , y , z components of vehicle and satellite position (see Eq. (5b)). Since the satellite position is not a function of the variables q_i , then $\frac{\partial \bar{\mathbf{S}}}{\partial q_i} = 0$, and the partial derivatives involving terms in $\bar{\mathbf{S}}$ need not be considered. Thus

$$\frac{\partial r_x}{\partial q_i} = \frac{\partial P_x}{\partial q_i}, \text{ etc.}$$

We may now write

$$\frac{\partial \mathbf{r}}{\partial q_i} = \left(\frac{\partial \mathbf{r}}{\partial r_x} \frac{\partial \mathbf{r}}{\partial r_y} \frac{\partial \mathbf{r}}{\partial r_z} \right) \begin{bmatrix} \frac{\partial P_x}{\partial q_i} \\ \frac{\partial P_y}{\partial q_i} \\ \frac{\partial P_z}{\partial q_i} \end{bmatrix}$$

The vehicle position at time t_n and, hence, \bar{P}_n is a function of the q_i variables φ_1 , λ_1 , A , and V at time t_1 . Moreover, the components of \bar{P}_n are expressed in terms of the inertial coordinates φ_n and θ_n . There is a simple coordinate transformation from these inertial coordinates to the rotating earth-fixed coordinates φ_n and λ_n . It is easily determined that

$$\frac{\partial \theta_n}{\partial q_i} = \frac{\partial \lambda_n}{\partial q_i}$$

and φ_n is identical in both systems. It is important to emphasize that the position coordinates φ_n and λ_n at time t_n are functions of the q_i parameters, φ_1 , λ_1 , A , and V at time t_1 .

Thus, the partial derivative coefficients for range are given in terms of their components as follows

$$\frac{\partial \mathbf{r}}{\partial q_i} = \left(\frac{\partial \mathbf{r}}{\partial r_x}, \frac{\partial \mathbf{r}}{\partial r_y}, \frac{\partial \mathbf{r}}{\partial r_z} \right) \begin{bmatrix} \frac{\partial P_x}{\partial \varphi} & \frac{\partial P_x}{\partial \theta} \\ \frac{\partial P_y}{\partial \varphi} & \frac{\partial P_y}{\partial \theta} \\ \frac{\partial P_z}{\partial \varphi} & \frac{\partial P_z}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \varphi_n}{\partial q_i} \\ \frac{\partial \lambda_n}{\partial q_i} \end{bmatrix}$$

Now the partial derivatives of range-rate \dot{r} are developed in piecemeal fashion. Again, the subscripts are dropped for the sake of generality.

From Eq. (6b) the partial derivatives for range rate are given by

$$\frac{\partial \dot{\bar{\mathbf{r}}}}{\partial q_i} = (\bar{\mathbf{v}}_P - \bar{\mathbf{v}}_S) \cdot \frac{\partial \left(\frac{\bar{\mathbf{r}}}{r} \right)}{\partial q_i} + \frac{\bar{\mathbf{r}}}{r} \cdot \frac{\partial (\bar{\mathbf{v}}_P - \bar{\mathbf{v}}_S)}{\partial q_i}$$

where $\bar{\mathbf{v}}_P$ and $\bar{\mathbf{v}}_S$ are the vehicle and satellite velocity vectors, respectively, $\bar{\mathbf{r}}$ is the range vector from the satellite to the vehicle, and r is the magnitude of this vector. On further breakdown note that

$$\begin{aligned} \frac{\partial \left(\frac{\bar{\mathbf{r}}}{r} \right)}{\partial q_i} &= \frac{\bar{\mathbf{r}}}{r} \frac{\partial \left(\frac{1}{r} \right)}{\partial q_i} + \frac{1}{r} \frac{\partial (\bar{\mathbf{r}})}{\partial q_i} \\ &= - \frac{\bar{\mathbf{r}}}{r^2} \frac{\partial r}{\partial q_i} + \frac{1}{r} \frac{\partial (\bar{\mathbf{r}})}{\partial q_i} \end{aligned}$$

The partial derivative for range rate may be written as follows

$$\begin{aligned} \frac{\partial \dot{\bar{\mathbf{r}}}}{\partial q_i} &= \left[- \frac{(\bar{\mathbf{v}}_P - \bar{\mathbf{v}}_S) \cdot \bar{\mathbf{r}}}{r^2} \frac{\partial r}{\partial q_i} \right] \quad \text{Term \#1} \\ &+ \left[\frac{(\bar{\mathbf{v}}_P - \bar{\mathbf{v}}_S)}{r} \cdot \frac{\partial (\bar{\mathbf{r}})}{\partial q_i} \right] \quad \text{Term \#2} \\ &+ \left[\frac{\bar{\mathbf{r}}}{r} \cdot \frac{\partial (\bar{\mathbf{v}}_P - \bar{\mathbf{v}}_S)}{\partial q_i} \right] \quad \text{Term \#3} \end{aligned}$$

Examining Term #1, note that the partial derivatives have already been determined. Therefore, we may write

$$\text{Term \#1} = -\frac{1}{r^2} \left[(V_{Px} - V_{Sx}), (V_{Py} - V_{Sy}), (V_{Pz} - V_{Sz}) \right] \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \frac{\partial r}{\partial q_i}$$

Considering Term #2, the partial derivatives are $\frac{\partial(\bar{r})}{\partial q_i}$. Using preceding results for $\frac{\partial r}{\partial q_i}$, we may determine that

$$\begin{bmatrix} \frac{\partial r_x}{\partial q_i} \\ \frac{\partial r_y}{\partial q_i} \\ \frac{\partial r_z}{\partial q_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial P_x}{\partial \varphi} & \frac{\partial P_x}{\partial \theta} \\ \frac{\partial P_y}{\partial \varphi} & \frac{\partial P_y}{\partial \theta} \\ \frac{\partial P_z}{\partial \varphi} & \frac{\partial P_z}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \varphi}{\partial q_i} \\ \frac{\partial \lambda}{\partial q_i} \end{bmatrix}$$

Term #2 can now be written as

$$\text{Term \#2} = \frac{1}{r} \left[(V_{Px} - V_{Sx}), (V_{Py} - V_{Sy}), (V_{Pz} - V_{Sz}) \right] \begin{bmatrix} \frac{\partial P_x}{\partial \varphi} & \frac{\partial P_x}{\partial \theta} \\ \frac{\partial P_y}{\partial \varphi} & \frac{\partial P_y}{\partial \theta} \\ \frac{\partial P_z}{\partial \varphi} & \frac{\partial P_z}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial \varphi}{\partial q_i} \\ \frac{\partial \lambda}{\partial q_i} \end{bmatrix}$$

Term #3 involves partial derivatives of the velocity expressions. Again note that the satellite velocity is not involved here since it is not a function of the q_i parameters, φ_1 , λ_1 , A , V . Only partial derivatives of the expressions for vehicle velocity need be considered.

Moreover, for the purposes of this computation, changes in vehicle altitude are neglected as they are not considered to be of practical significance. The vehicle velocity is a direct function of all four parameters which may in turn be functions of each other. Therefore,

$$\begin{bmatrix} \frac{\partial V_{Px}}{\partial q_i} \\ \frac{\partial V_{Py}}{\partial q_i} \\ \frac{\partial V_{Pz}}{\partial q_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial V_{Px}}{\partial \phi} & \frac{\partial V_{Px}}{\partial \theta} & \frac{\partial V_{Px}}{\partial A} & \frac{\partial V_{Px}}{\partial V} \\ \frac{\partial V_{Py}}{\partial \phi} & \frac{\partial V_{Py}}{\partial \theta} & \frac{\partial V_{Py}}{\partial A} & \frac{\partial V_{Py}}{\partial V} \\ \frac{\partial V_{Pz}}{\partial \phi} & \frac{\partial V_{Pz}}{\partial \theta} & \frac{\partial V_{Pz}}{\partial A} & \frac{\partial V_{Pz}}{\partial V} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial q_i} \\ \frac{\partial \lambda}{\partial q_i} \\ \frac{\partial A}{\partial q_i} \\ \frac{\partial V}{\partial q_i} \end{bmatrix}$$

Now Term #3 is written in terms of its components

$$\text{Term \#3} = \frac{1}{r} (r_x, r_y, r_z) \begin{bmatrix} \frac{\partial V_{Px}}{\partial \phi} & \frac{\partial V_{Px}}{\partial \theta} & \frac{\partial V_{Px}}{\partial A} & \frac{\partial V_{Px}}{\partial V} \\ \frac{\partial V_{Py}}{\partial \phi} & \frac{\partial V_{Py}}{\partial \theta} & \frac{\partial V_{Py}}{\partial A} & \frac{\partial V_{Py}}{\partial V} \\ \frac{\partial V_{Pz}}{\partial \phi} & \frac{\partial V_{Pz}}{\partial \theta} & \frac{\partial V_{Pz}}{\partial A} & \frac{\partial V_{Pz}}{\partial V} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial q_i} \\ \frac{\partial \lambda}{\partial q_i} \\ \frac{\partial A}{\partial q_i} \\ \frac{\partial V}{\partial q_i} \end{bmatrix}$$

Having formed the three terms it is a simple matter to combine them to express the partial derivatives of range rates in terms of their components.

Partial Derivatives of Range and Range Rate

Partial derivatives of range with respect to the components of \bar{r} :

$$\frac{\partial r}{\partial r_x} = \frac{r_x}{r}$$

$$\frac{\partial r}{\partial r_y} = \frac{r_y}{r}$$

$$\frac{\partial r}{\partial r_z} = \frac{r_z}{r}$$

Partial derivatives of the components of \bar{P} with respect to φ and θ :

(Note again that $|\bar{P}| = P$ is considered constant.)

$$\frac{\partial P_x}{\partial \varphi} = P(-\sin \varphi \cos \theta)$$

$$\frac{\partial P_x}{\partial \theta} = P(-\cos \varphi \sin \theta)$$

$$\frac{\partial P_y}{\partial \varphi} = P(-\sin \varphi \sin \theta)$$

$$\frac{\partial P_y}{\partial \theta} = P(\cos \varphi \cos \theta)$$

$$\frac{\partial P_z}{\partial \varphi} = P \cos \varphi$$

$$\frac{\partial P_z}{\partial \theta} = 0$$

Partial derivatives of the components of \bar{V}_p with respect to φ :

$$\frac{\partial V_{Px}}{\partial \varphi} = V(-\cos A \cos \theta \cos \varphi) + Pn_E(\sin \theta \sin \varphi)$$

$$\frac{\partial V_{Py}}{\partial \varphi} = V(-\cos A \sin \theta \cos \varphi) - Pn_E(\sin \varphi \cos \theta)$$

$$\frac{\partial V_{Pz}}{\partial \varphi} = V(-\cos A \sin \varphi)$$

Partial derivatives of the components of \bar{V}_P with respect to θ :

$$\frac{\partial V_{Px}}{\partial \theta} = V(\cos A \sin \theta \sin \varphi - \sin A \cos \theta) - Pn_E(\cos \theta \cos \varphi)$$

$$\frac{\partial V_{Py}}{\partial \theta} = V(-\cos A \cos \theta \sin \varphi - \sin A \sin \theta) - Pn_E(\cos \varphi \sin \theta)$$

$$\frac{\partial V_{Pz}}{\partial \theta} = 0$$

Partial derivatives of the components of \bar{V}_P with respect to A :

$$\frac{\partial V_{Px}}{\partial A} = V(\sin A \cos \theta \sin \varphi - \cos A \sin \theta)$$

$$\frac{\partial V_{Py}}{\partial A} = V(\sin A \sin \theta \sin \varphi + \cos A \cos \theta)$$

$$\frac{\partial V_{Pz}}{\partial A} = V(-\sin A \cos \varphi)$$

Partial derivatives of the components of \bar{V}_P with respect to V :

$$\frac{\partial V_{Px}}{\partial V} = (-\cos A \cos \theta \sin \varphi - \sin A \sin \theta)$$

$$\frac{\partial V_{Py}}{\partial V} = (-\cos A \sin \theta \sin \varphi + \sin A \cos \theta)$$

$$\frac{\partial V_{Pz}}{\partial V} = (\cos A \cos \varphi)$$

(Note that in the components below, the subscript notation indicates the value at time t_n .)

Partial derivatives of φ_n with respect to $\varphi_1, \lambda_1, A, V$:

$$\frac{\partial \varphi_n}{\partial \varphi_1} = 1, \quad \frac{\partial \varphi_n}{\partial \lambda_1} = 0 \quad n = 1, 2, 3, \dots$$

$$\frac{\partial \varphi_n}{\partial A} = \frac{\partial \varphi_n}{\partial V} = 0 \quad n = 1$$

$$\frac{\partial \varphi_n}{\partial A} = -\frac{V(t_n - t_1)}{P} \sin A, \quad \frac{\partial \varphi_n}{\partial V} = \frac{(t_n - t_1)}{P} \cos A \quad n = 2, 3, 4, \dots$$

Partial derivatives of λ_n with respect to $\varphi_1, \lambda_1, A, V$:

$$\frac{\partial \lambda_n}{\partial \lambda_1} = 1 \quad n = 1, 2, 3, \dots$$

$$\frac{\partial \lambda_n}{\partial \varphi_1} = \frac{\partial \lambda_n}{\partial A} = \frac{\partial \lambda_n}{\partial V} = 0 \quad n = 1$$

$$\frac{\partial \lambda_n}{\partial \varphi_1} = \tan A (\sec \varphi_n - \sec \varphi_1) \quad n = 2, 3, 4, \dots$$

$$\frac{\partial \lambda_n}{\partial A} = \frac{\lambda_n - \lambda_1}{\sin A \cos A} + \tan A \sec \varphi_n \frac{\partial \varphi_n}{\partial A} \quad n = 2, 3, 4, \dots$$

$$\frac{\partial \lambda_n}{\partial V} = \tan A \sec \varphi_n \frac{\partial \varphi_n}{\partial V} \quad n = 2, 3, 4, \dots$$

Partial derivatives of A with respect to φ_1 , λ_1 , A, V:

$$\frac{\partial A}{\partial A} = 1$$

$$\frac{\partial A}{\partial \varphi_1} = \frac{\partial A}{\partial \lambda_1} = \frac{\partial A}{\partial V} = 0$$

Partial derivatives of V with respect to φ_1 , λ_1 , A, V:

$$\frac{\partial V}{\partial V} = 1$$

$$\frac{\partial V}{\partial \varphi_1} = \frac{\partial V}{\partial \lambda_1} = \frac{\partial V}{\partial A} = 0$$

REFERENCES

1. Campbell, J. W., Predicted Operational Requirements for a Non-Military Traffic Coordination and Navigation Satellite System, Technology Audit Corporation, Washington, D. C., TAC No. 107-10, undated.
2. Study of Satellites for Navigation, The General Electric Company, February 1964.
3. Autoscan, An Automatic Satellite/Computer Aid to Navigation, Univac, Blue Bell, Pennsylvania, Contract No. NASW-706, February 1, 1964.
4. Interim Study Report on the Navigation/Control Satellite System, Philco Corporation, Western Development Laboratories, Palo Alto, California, WDL-B279, July 24, 1964.
5. Navigation Satellite System, Vol. 1, Westinghouse Electric Corporation, Defense and Space Center, Baltimore, Maryland, January 30, 1964.
6. Myers, H. A., "Aircraft Navigation by Satellite," J. Inst. Nav., Vol. 16, No. 4, October 1963.
7. Ridenour, L. N. (ed.), Radar System Engineering, Vol. 1, Chapter V, McGraw-Hill Book Company, Inc., New York, 1947, p. 154.
8. Kelly, E. J., I. S. Reed, and W. L. Root, "The Detection of Radar Echoes in Noise," J. Soc. Indust. Appl. Math., Vol. 8, No. 2, June 1960.
9. Kelly, E. J., The Radar Measurement of Range, Velocity, and Acceleration, Massachusetts Institute of Technology, Lincoln Laboratory, Lexington, Massachusetts, January 6, 1961.
10. Manasse, R., Parameter Estimation Theory and Some Applications of the Theory to Radar Measurements, The Mitre Corporation, Technical Series Report No. 3, undated.
11. Manasse, R., Summary of Maximum Theoretical Accuracy of Radar Measurements, The Mitre Corporation, Technical Series Report No. 2, undated.
12. Brennan, L. E., Angle Tracking Accuracy of Phased Array Radars, The RAND Corporation, RM-3587-ARPA, April 1963.
13. Skolnik, M. I., Introduction to Radar Systems, McGraw-Hill Book Company, Inc., New York, 1962.

14. Bellman, R., R. Kalaba, and J. Lockett, Dynamic Programming and Ill-Conditioned Linear Systems-II, The RAND Corporation, RM-4201-PR, July 1964.
15. Bellman, R., R. Kalaba, and J. Lockett, Dynamic Programming and Ill-Conditioned Systems, The RAND Corporation, RM-3815-PR, December 1963.
16. Moulton, F. R., An Introduction to Celestial Mechanics, 2d ed., MacMillan Company, Inc., New York, 1914.